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# Editorial

In this edition of our journal, we feature a compelling research paper that delves into the application of spectral methods based on classical orthogonal polynomials for solving digital information processing problems. This study highlights innovative approaches to signal approximation and noise filtering, providing significant contributions to the field of digital signal processing.

The featured paper explores the use of spectral methods grounded in classical orthogonal polynomials to tackle various challenges in digital information processing. The research is divided into two primary areas: signal approximation using Jacobi polynomials and noise filtering using Chebyshev-Laguerre polynomials [1].

This edition's featured paper exemplifies the type of innovative and impactful research that our journal strives to publish. By leveraging classical orthogonal polynomials, the study offers novel solutions for signal approximation and noise filtering in digital information processing. We are excited to share these insights with our readers and anticipate that they will inspire further advancements and research in the field.

**References:**

[1] Y. Pyanylo, V. Sobko, H. Pyanylo, O. Pyanylo, “Orthogonal Polynomials in the Problems of Digital Information Processing,” Journal of Engineering Research and Sciences, vol. 2, no. 5, pp. 1–9, 2023, doi:10.55708/js0205001.

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# CONTENTS

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# Orthogonal Polynomials in the Problems of Digital Information Processing

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**ABSTRACT:** The paper examines spectral methods based on classical orthogonal polynomials for solving problems of digital information processing. Based on Jacobi polynomials, signal approximation methods are built to identify objects in the natural environment. Based on Chebyshev-Laguerre polynomials, methods of filtering multiplicative signal noises in linear filter models are proposed. Numerical experiments on model problems were conducted.

**KEYWORDS:** Spectral Methods, Digital Information Processing, Signal Approximation and Filtering, Object Identification

## 1. Introduction

To solve a broad spectrum of scientific and applied problems in many areas of natural science and technology, we use information that is stored in digital form. Input information is the value of measured parameters of the processes under investigation. The accuracy of the measured values depends on the measuring equipment and is not always high. Such problems as remote sensing data processing, electronic paramagnetic resonance (EPR), tomography, etc. [1] can be categorized as problems with unclear input information. Digital information is considered input for developing mathematical models of physical processes. Since the input information is specified with the inaccuracy of the measuring devices, their direct use in mathematical models can lead to significant differences in the desired results [2]. Consequently, preliminary processing of the input information is required [3, 4].

It becomes necessary to choose the processing method for cases when solutions to formulated applied problems that are based on digital input data should be obtained [3, 5, 6]. Considerable amount of software packages has been developed to process digital information. [7, 8]

The total amount of worldwide commercial software applications is measured by several dozen. However, the number of the most famous and widely used commercial software applications for geoinformation systems can be limited to ten to fifteen [7, 8]. The world leaders of the software are ESRI packages (ArcGIS Desktop products), the MapInfo Professional package, the Idrisi package (developed at Clark University, USA), AutoCAD, ERDAS IMAGINE, ENVI, and Digitals [4].

Every process in which digital information is used has its peculiarities. These peculiarities cannot always be factored into the existing application programs. As a rule, these software packages are protected and do not allow any modifications In this regard, it is necessary to develop methods of processing input data that should make it possible to consider the particularity that occurs when exploring numerous natural phenomena.

Signal approximation, information compression, and signal filtering are the main problems in information processing [1]. There are various methods to solve data processing problems. Not all of them meet the requirements associated with the tasks that must be resolved. Spectral methods in orthogonal bases with fast transformation algorithms are the most used. The analysis of the orthogonal bases presented in the literature shows that the classic orthogonal polynomials Jacobi and Chebyshev-Laguerre are highly effective for usage. [2, 3, 5, 6, 9 - 12].

The paper is aimed at research methods of classical orthogonal polynomials usage for solving applied problems, in particular, object identification based on the processing of discrete data of electronic paramagnetic resonance (EPR-dosimetry), remote sensing, and others. The identification of objects takes place based on the characteristic signs of the separation of the abnormal spectrum from the background spectrum. These signs may be extreme values of the digital spectrum. At the same time, not only the extreme values themselves are significant, but their coordinates as well.

It is worth noting, that digital information processing problems are also widely used in the mathematical modeling of natural and physical processes. In particular, for mass transfer modeling in complex pipeline systems and the filtration process in complex porous media. The peculiarity of such problems is that the input information is measured data. These data are obtained with low accuracy and at arbitrarily placed points. Direct implementation of such data in mathematical models can lead to unacceptable results in practice. Preliminary statistical and probabilistic data processing is necessary for effective input information implementation.

Adaptive algorithms based on classical orthogonal polynomials for solving many applied problems, such as approximation and filtering of signals, solving problems of mathematical physics, etc., are built in the current manuscript. The advantage of these algorithms over the existing ones is that these methods ensure the accuracy of the recovery of the problem solution based on the so-called constructive elements.

## 2. Definition of integral transformations where Jacobi and Chebyshev-Laguerre polynomials are cores.

A characteristic feature of the applied problems is that, in most cases, the data needed to set the initial and boundary conditions in an analytical form are unknown. This leads to the formulation of incorrect problems of mathematical physics under great uncertainty. To build a solution to problems of this type, it is necessary to use regularizing algorithms based on a priori information that is known in a non-formalized form.

One of the forward-looking approaches for resolving the formulated problems is operational and spectral methods usage. In computational experiments, these methods have several advantages related to the parametric (symbolic) problems solution at different stages. Such solutions allow us to study the influence of the parameters of the physical process on its progress and the stability of the solution regarding small perturbations of the input data. In those cases when the symbolic solution of the problem cannot be obtained, the value of these methods is partially retained, since it is possible to build a solution based on the images (in terms of the corresponding operators) based on the so-called constructive elements, for these cases the values of the images are then calculated with a predetermined accuracy.

Constructive elements are functions specified in some functional basis by a finite set of integers or rational numbers with accuracy up to the normalizing factor. This class of elements includes, for example, classical orthogonal polynomials; each such element has integer coefficients. Using the basis of structural elements allows you to organize the computational process of solving the problem with guaranteed accuracy. The calculation with guaranteed accuracy means the ability to build the calculation process in a way that arithmetic operations errors at each intermediate stage of the machine (hardware) implementation have such an order as to ensure the sufficient accuracy of the final result.

When numerically finding a solution to a problem with guaranteed accuracy, mainly method errors are left. If the numerical methods ensure sufficient accuracy of the problem solution recovery, then it can be assumed that the advantages of the symbolic solution in the computational experiment are maintained.

Let the function $\varphi(x)$ be defined on the interval of $[-1,1]$, and meet the conditions which allow it to be expanded into a series by Jacobi polynomials. [12, 13]. The Jacobian integral transformation $\varphi_n$ of $\varphi(x)$ function with the weighting function $\omega(x) = (1-x)^{\alpha}(1+x)^{\beta}$ $(\alpha > -1, \beta > -1)$ is defined as follows

$$\varphi_n = \int_{-1}^{1} \omega(x)\, P_n^{(\alpha,\beta)}(x)\, \varphi(x)\, dx\ . \qquad (1)$$

where $P_n^{(\alpha,\beta)}(x)$ – are Jacobi polynomials. If the images $\varphi_n$ for $n = 0,1,2\ldots$, are known, then the inverse of the integral transformation (1) is found by the formula

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{\varphi_n}{r_n} P_n^{(\alpha,\beta)}(x)\ . \qquad (2)$$

For a function $f(t)$, $t \in [0,\infty)$, that expands in a series by $L_m^{\lambda}(t), \lambda > -1$ Chebyshev-Laguerre polynomials, we define the corresponding integral transformation as follows

$$f_m = \int_0^{\infty} t^{\lambda} e^{-t} L_m^{\lambda}(t) f(t) dt\ . \qquad (3)$$

In this case, $f_m$ is the Chebyshev-Laguerre image of the original function $f(t)$. The formula for the integral transformation inversion (3) has the form

$$f(t) = \sum_{m=o}^{\infty} \frac{m! f_m}{\Gamma(m+\lambda+1)} L_m^{\lambda}(t)\ . \qquad (4)$$

Analysis of the spectral methods implementation shows that the basis used to solve the problem must correlate with the domain of definition. That means that in

a case where the studied parameter changes over a finite interval, then Jacobi polynomials are advisable to use as this interval is reduced to an orthogonality interval by linear substitution. In the case of a semi-infinite interval, Chebyshev-Laguerre polynomials are recommended to be used. Chebyshev-Laguerre polynomials grow exponentially over increasing an argument. This property complicates the summation of the series (4). One of the methods to solve this problem is to use the modified Chebyshev-Laguerre integral transformation.

Let $\nu$ $(\nu < 0, \nu \neq 0)$ and $\mu(0 < \mu < \infty)$ are some constants. The modified integral Chebyshev-Laguerre transform is defined by the relation [5]

$$f_m(\nu,\mu) = \int_0^\infty t^{\nu\lambda+\nu-1} e^{-\mu t^\nu} L_m^\lambda(\mu t^\nu) f(t) dt \ . \quad (5)$$

For such an integral transformation, the original $f(t)$ is found by the formula

$$f(t) = \sum_{m=o}^{\infty} \frac{m! f_m(\nu,\mu)}{\Gamma(m+\lambda+1)} L_m^\lambda(\mu t^\nu) \ . \quad (6)$$

We note that the integral transformation (5) with the appropriate choice of parameters allows us to study rapidly oscillating functions effectively.

Spectral methods on a biorthogonal basis [11, 14], built based on classical orthogonal polynomials, have proved their worth. Their use makes it possible to apply the implicit regularization of the corresponding series to a greater extent.

Integral transformations (1) and (3) or similar ones are applied to solve many applied problems. They are the basis of spectral methods of digital information processing. The computation of the coefficients of the orthogonal series (integral transformations (1), (3), (6)) and finding the corresponding originals is the main essence of the spectral methods.

Quadrature formulas for calculating the integral transformations, the method of least squares, or other methods can be used to find generalized spectra when input values are given in the discrete form. [15, 16]. In some cases, depending on the method of specifying input information, it is possible to determine formulas that are optimal in the $L_2$ class for the calculation of the generalized spectra.

Let the polynomials $u_n(x)$ be orthogonal on the interval $[a, b]$ and the function $\varphi(x)$ is presented by an orthogonal series over the polynomials $u_n\ (x)$ [5], i.e

$$\varphi(x) \ = \sum_{n=0}^{\infty} \frac{\varphi_n}{r_n} \ u_n \ (x) \ . \quad (7)$$

It is known that the $N+1$-th orthogonal polynomial has $N+1$ a real root that belongs to the interval of orthogonality. Then an $L_2$ optimal quadrature formula is used to calculate the generalized spectra $\varphi_n$

$$\varphi_n \ \sim \sum_{j=0}^{N} \rho_j^2 \ u_n \ (x_j) \ \varphi(x_j) \ , \quad (8)$$

where $x_j$ - the roots of the equation $u_{N+1}(x_j) = 0$, a

$$\rho_j^{-2} \ = \ \sum_{i=0}^{N} u_i^2 \ (x_j).$$

## 3. Construction of biorthogonal systems of functions.

The advisability of constructing a biorthogonal basis may explain the following well-known fact: when a biorthogonal series converges slowly by one system of functions then it converges rapidly by another. Nowadays, there are not many works in literature that are devoted to the construction of biorthogonal systems of functions and the studying of their characteristics. The main ways of constructing an orthogonal basis are: using the Gram-Schmidt scheme, building eigenvalues and eigenfunctions, and building eigenvalues of integral operators. It must be taken into account that, in practice, the Gram-Schmidt scheme permits the construction of a small number of orthogonal functions as it comes down to solving a poorly conditioned system of algebraic equations.

The systems of the orthogonal polynomials $p_0, p_1, \cdots p_n$ , which are built based on tram-Schmidt's scheme, rely on the following algorithm. Let $g_n(x) = x^n$ be a polynomials system on the $x\epsilon[x_1, x_k]$ interval. Let's define a projector

$$proj_p(g) = \frac{\int_{x_1}^{x_2} f(x)g(x)W(x)dx}{\int_{x_1}^{x_2} (f(x))^2 W(x)dx}.$$

Then orthogonal polynomials are determined sequentially according to the scheme

$$p_0 = g_0,$$
$$p_1 = g_1 - proj_{p_0}(g_1),$$
$$p_k = g_k - \sum_{j=1}^{k-1} proj_{p_j}(g_k).$$

The given algorithm for constructing orthogonal polynomials belongs to the unstable class. The errors of decomposing rounding and numerical integration rapidly increase as the polynomial error increases during coefficient calculation.

Construction of quasi-spectral polynomials based on Chebyshev polynomials $T_n(x)$ . Consider an integral operator $L: L_{2,\rho}[-1,1] \to L_{2,\rho}[-1,1]$ with a weight function $\rho = 1/\sqrt{1-x^2}$, which matches the expression $f \in [-1,1]$ for the function

$$Lf(x) = \int_{-1}^{x} dx_1 \int_{-1}^{x_1} f(x_2)dx_2 = \int_{-1}^{x} (x - x_1) f(x_1)dx_1 \ (9)$$

It is known [12, 13, 17] that expression (9) has no non-zero eigenvalues. Therefore, we will consider the

corresponding quasi-spectral problem, based on the properties of the integral operator (9) in the Hilbert space $L_{2,\rho}[-1,1]$ instead of the spectral eigenvalue problem. Let's construct the operator

$$\pi_1^{\infty} L = \pi_1^{\infty} \int_{-1}^{x} \int_{-1}^{x_1} \quad : \pi_1^{\infty} \int_{-1}^{x} \int_{-1}^{x_1} \quad ,$$

which converts odd polynomials into even ones, and vice versa.

Definitions. Let $\tilde{L}_{2,1}[-1,1]$ is a complete subspace of Hilbert space $L_{2,\rho}[-1,1]$, all elements of which satisfy the $\int_{-1}^{1} u(x)dx = 0$ integral equation. The values of the $\pi_1^{\infty} \int_{-1}^{x} \int_{-1}^{x_1} \quad : \tilde{L}_{2,1}[-1,1] \to \tilde{L}_{2,1}[-1,1]$ operator, based on Chebyshev polynomials elements, differ from the values of the operator (1) by a zero summand of the Fourier-Chebyshev series. However, the spectral characteristics of these operators are radically different. The operator $\pi_1^{\infty} \int_{-1}^{x} \int_{-1}^{x_1}$ has characteristic values and eigenfunctions, but the spectral radius of a compact operator of the Volterra type has no eigenvalues that differ from zero [12, 13, 17].

Statement 1. If a polynomial

$$\bar{U}_{2i-1}^{2s-1}(x) = \sum_{j=1}^{s} \bar{c}_{2j-1}^{2i-1} T_{2j-1}(x),\ i = 1,\dots,s$$

meets the conditions

$$\int_{-1}^{1} \rho(x) U_{2i-1}^{2s-1}(x) \bar{U}_{2j-1}^{2s-1}(x) = \begin{cases} 0, i \neq j \\ \sigma_{2i-1}, i = j' \end{cases}$$

then the coefficients $c_{2j-1}^{2i-1}$ are computed according to the algorithm that is constructed in the works [4, 17].
Statement 2. If a polynomial

$$\bar{U}_{2i}^{2s}(x) = \sum_{j=1}^{s} \bar{c}_{2j}^{2i} T_{2j}(x),\ i = 1,\dots,s$$

meets the conditions

$$\int_{-1}^{1} \rho(x) U_{2i}^{2s}(x) \bar{U}_{2j}^{2s}(x) = \begin{cases} 0, i \neq j \\ \sigma_{2i}, i = j' \end{cases}$$

then its coefficients $c_{2j}^{2i}$ are computed by the algorithm presented in the works [4, 14].

Construction of quasi-spectral polynomials based on Lager polynomials. Consider an integration operator $L: L_{2,\varpi}[-1,1] \to L_{2,\varpi}[-1,1]$ with a weight function $\varpi(x) = \exp(-t)$, that $f(t) \in L_{2,\varpi}[0,\infty]$ matches the expression

$$Lf(t) = \int_{-1}^{t} dt_1 \int_{-1}^{t_1} f(t_2)dt_2 = \int_{-1}^{t} (t - t_1) f(t_1)dt_1 \quad (10)$$

Expression (10) does not have non-zero eigenvalues. Let's consider the corresponding quasi-spectral problem.

The quasi-spectral problem for an integral operator. For given $n = 1,2,\dots$ we should find the values of the parameter $\lambda$ at which the equation

$$\int_0^t dt_1 \int_0^{t_1} \varphi(t_2)\, dt_2 = -\lambda \left( \varphi(t) - \int_0^t \varphi(t_1) dt_1 \right) + \\ + \tau_1 L_{n+1}(t) + \tau_0 L_0(t)$$

has nonzero polynomial solutions $\varphi = \varphi(t) \in L_{2,\varpi}[0,\infty]$ of degree $\leq n$, where $\tau_1$, $\tau_0$ are some parameters, $L_{n+1} = L_{n+1}(t)$ and $L_0 = L_0(t)$ are given (and fixed) Lager's polynomials of power that equal to $n+1$ and zero, respectively.

The following equations are veritable

$$\int_0^{\infty} \exp(-t) \varphi_i^{n-1}(t) \varphi_j^{n-1}(t)\, dt = \begin{cases} 0, i \neq j \\ 1, i = j' \end{cases}$$

$$\int_0^{\infty} \exp(-t) \psi_i^{n}(t) \psi_j^{n}(t) dt = \begin{cases} 0, i \neq j \\ -\lambda_i, i = j \end{cases}.$$

Based on the constructed biorthogonal polynomials, adaptive algorithms for the computation of mode parameters of gas transport were built. These algorithms permit minimizing energy resources and optimizing the operation of the gas station according to various criteria. The fact that a significant number of parameters can be calculated with arbitrary precision and stored in computer memory is the advantage of using biorthogonal polynomials constructed to solve problems of mathematical physics. Their further usage for problem calculations reduces the accumulation of errors and saves time.

## 4. Moments in the processing of digital information and their relationship with spectral methods

Spectral moments in statistical and probabilistic methods of digital information processing should be used to reduce the error values that occur under the approach of the argument of the distribution function to the initial value. This error can occur by using distribution moments instead of the distribution function to simplify the analysis [1, 3, 4, 7, 11].

In contradistinction to the moments of the distribution, the spectral moments of the distribution do not depend on the shift of the distribution function along the axis of the argument. They can be initial and central, just as distribution moments. But in most cases, the spectral moments to the distribution functions are central [18].

The introduction of the concept of spectral distribution moments will simplify the computation of several errors summing up during their quantile evaluation [5, 8, 12-17, 19].

The initial moment of the k-th order of a random variable X is called the mathematical expectation of the k-th degree X, i.e.

$$M_k(X) = \int_a^b x^k f(x)\, dx = a_k,$$

The central point of the k-th order is called the mathematical expectation of the k-th degree of random variable deviation from its mathematical expectation, i.e.

$$\mu_k = \int_a^b (x - a_1)^k f(x)\, dx$$

The central moment of the first order is zero; the central moment of the second order is equal to the variance of the random variable: μ1 = 0; μ2 = D (X). The central moments are used to quantify the distribution of a random variable.

The asymmetry coefficient $C_s$ is the ratio of the central moment of the third order to the cube of the standard deviation

$$C_s = \frac{\mu_3}{\mu_2^{3/2}}.$$

The kurtosis coefficient of the random variables is calculated by the formula

$$E_s = \frac{\mu_4}{\mu_2^2} - 3.$$

Using orthogonal distributions permits solving some problems of digital information processing, for example, signal approximation and filtering from additive noise. At the same time, by the known coefficients, using the relationship between the coefficients of orthogonal distributions and statistical parameters, it is possible to estimate their probability and other probabilistic-statistical parameters.

Consider the relationship of moments with generalized spectral coefficients based on Jacobi polynomials.

If the function $\varphi(x)$ is decomposed into series

$$\varphi(x) = \omega(x) \sum_{n=0}^{\infty} \frac{\varphi_n}{r_n} P_n^{(\alpha,\beta)}(x),$$

then

$$\varphi_n = \int_{-1}^{1} \varphi(x)\, P_n^{(\alpha,\beta)}(x)\, dx .$$

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are given in the form of

$$P_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \eta_{jn}(\alpha,\beta) x^j.$$

If through

$$\mu_j = \int_{-1}^{1} \varphi(x)\, x^j\, dx$$

denote the moments of the function $\varphi(x)$, then

$$\varphi_n = \sum_{j=0}^{n} \eta_{jn}(\alpha,\beta)\mu_j.$$

The last formula allows us to find the moments of the function for $n$ an arbitrary spectral coefficient $\varphi_n$

$$\mu_n = \frac{1}{\eta_{nn}(\alpha,\beta)}\left(\varphi_n - \sum_{j=0}^{n-1} \eta_{jn}(\alpha,\beta)\mu_j\right).$$

Below is an example of the implementation of statistical and probabilistic data processing. The works [2. 10, 11] are related to the mathematical model construction of the process of the Ukraine gas transportation system operation together with the underground gas storage operation. The input information is the measured pressure values and volumetric gas consumption at the measurement points. Based on this data, a range of practical problems are solved. In particular, the problem of determining gas reserves in pipelines can be solved. The GTS mostly operates in stationary mode. The calculation results of the gas reserve changes in the 122 km long and the 1,388 m diameter pipeline are presented in Figure 1. These results are calculated based on the measured data of the inlet and outlet pressures every two hours.

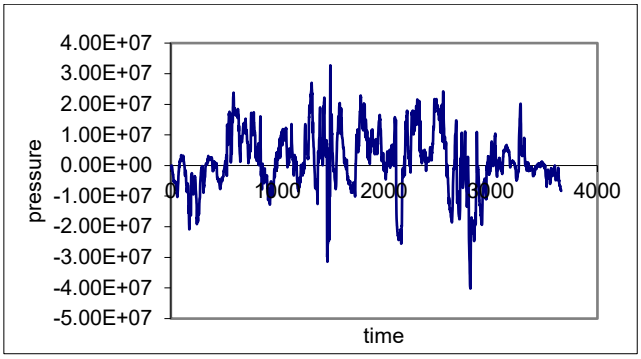


Figure 1: Dependency of gas reserves changes in the 122 km long and the 1,388 m diameter pipe, where calculations are based on measured data of inlet and outlet pressures every two hours.

The implementation of statistical and probabilistic input data processing allows stabilizing the process of the gas volume determination in the pipeline and determining the necessary processing parameters: the number of measurements, time interval, etc.

Identifying the nature of gas movement in the pipeline, stationary or non-stationary, is another objective, which is solved using the Fourier-Jacobi series. The zero Fourier-Legendre coefficient determines the arithmetic average of the measured value in the corresponding interval. The first coefficient expresses the mathematical expectation of a random value. Based on an analysis of the numerical results, we can follow up: if the first Fourier-Jacobi coefficient is smaller than one, then the gas movement process is stable; otherwise - it is non-stationary.

## 5. Inversion of one-dimensional Volterra convolution

Obtaining digital information about the explored objects is associated with various types of inaccuracies - additive or multiplicative. Additive inaccuracies are sufficiently well filtered by approximating the received signal. As a rule, filtering multiplicative noises is solved by integral equations, the cores of which are functions of the medium capacity. Quite often, integral equations may be approximated by convolutional integral equations

$$\alpha f(t) + \mu \int_0^t k'(t-\tau)\, f(\tau)\, d\tau = y(t), \qquad (11)$$

$$\mu \int_0^t k(t-\tau)\, f(\tau)\, d\tau = y(t). \qquad (12)$$

The problem of filtering multiplicative noises in linear filter models of signals, information processing, lidar equations, etc. may be solved by such equations [1, 5, 8, 20]. Here $\alpha$, $\mu$- some constants, $f(t)$- the desired function, $k(t)$- the equation core. Equations (11) and (12) can be easily solved by operational methods based on the integral Laplace transformation. When input information is presented as discrete data, then it is necessary to apply numerical methods. The main ways to solve integral equations of the 1st type convolution is the utilization of regularization algorithms of the Tikhonov type or the discretization of the direct origin equation. However, these methods have some disadvantages [2, 20].

The use of Tikhonov-type algorithms leads to a loss of Volterra stability, which significantly reduces the ability to restore the searched functions for the areas under consideration and demands the implementation of small grid steps.

The main disadvantage of the second method is the lack of accounting for the instability of the digital solution to inaccuracy in the input information. That leads to the case when the solution of the perturbed equation is beyond the set of correctness. Furthermore, not all quadrature formulas give rise to convergent methods.

Algorithm for solving integral equations (11) and (12) in the Chebyshev-Laguerre basis $L_n^{\lambda}(t)$, $\lambda > -1$. [5] It is assumed that the functions included in integral equations (11) and (12) satisfy the conditions that allow them to be represented by the Fourier-Laguerre series (4). Then, if $f(t)$ is the restoring signal, then the determination of the unknown coefficients $f_n$. will ensure the search solution.

Solution of equation (11). It is convenient to write equation (10) as follows

$$\big(\alpha - \mu\, k(0)\big)\, f(t) + \mu \frac{d}{dt} \int_0^t k(t-\tau)\, f(\tau)\, d\tau = y(t).$$

If $\lambda = 0$, then there is a formula for calculating the unknown coefficients

$$f_n = \frac{1}{\alpha + \mu(k_0 - k(0))} \left(y_n - \mu \sum_{m=1}^{n} k_m\, f_{n-m}\right). \qquad (13)$$

Here $k_m$ and $f_m$ are the Fourier-Laguerre coefficients of the functions $k(t)$ and $f(t)$, respectively. Since the coefficients $k_n$ and $y_n$ are known, the Lager spectrum of the unknown function $f(t)$.is determined by the formula (13). Thus, equation (11) can be considered resolved.

Similarly, the solution of equation (11) is obtained. Let the functions $k(t)$ and $f(t)$ be given in series by polynomials $L_n^{\lambda_k}(t), \lambda_k > -1$, and $L_n^{\lambda_f}(t), \lambda_f > -1$, respectively, then the Fourier-Laguerre coefficients for the desired solution are calculated by the formula

$$f_n = \frac{1}{k_0} \left(\frac{1}{\mu}\, y_n - \sum_{m=0}^{n-1} k_{n-m}\, f_m\right). \qquad (14)$$

Here $y_n$ are the Fourier-Laguerre coefficients of the function $y(t)$ at $\lambda = \lambda_k + \lambda_f + 1$.

One of the advantages of the considered method is that the discretization procedure is excluded since the integral convolution turns into a series convolution. Quadrature formulas and formulas (13) and (14) enable the computation of a limited number of coefficients. These are not always sufficient to restore the original with the required accuracy. For this disadvantage elimination, asymptotic formulas for large $n$ were derived in the research that applies a priori information. They allow building a regularizing algorithm for restoring the desired function $f(t)$ with an accuracy that is acceptable for practice.

Modification of the constructed scheme. It is advisable to modify the above spectral method of solving convolution-type integral equations in a way as to restore values $f(t)$ by values $y(t_i)$ at the given points. To do this, we write formula (14) in the form

$$y_n = \sum_{m=0}^{n} k_{n-m}\, f_m,$$

or in matrix form

$$K_N\, F_N = Y_N, \qquad (15),$$

where

$$K_N = \begin{pmatrix} k_0 & 0 & \dots & 0 \\ k_1 & k_0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ k_{N_1} & k_{N-2} & \dots & k_0 \end{pmatrix}, \quad F_N = \begin{pmatrix} f_0 \\ k_1 \\ \vdots \\ f_{N_1} \end{pmatrix}, \quad Y_N = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N_1} \end{pmatrix}.$$

The unknown coefficients $f_n$ will be the solutions of the matrix equation (14)

$$F_N = K_N^{-1}\, Y_N.$$

Here $K_N^{-1}$ is the matrix inverted to $K_N$. In expanded form

$$f_n = \sum_{i=0}^{N-1} y_i\, z_{n,i}, \quad n = \overline{0, N-1}. \qquad (16)$$

The last formula $z_{n,i}$ contains the elements of the inverse matrix$K_N^{-1}$. Substitute formula (14) into the $N$-th partial sum of the Fourier-Laguerre series. As a result, we get approximate equality for recovery $f(t) \approx f_N(t)$

$$f_N(t) = \sum_{n=0}^{N-1} \eta_n^{\lambda}(t)\, y_n, \qquad (17)$$

where

$$\eta_n^{\lambda}(t) = t^{\lambda} \sum_{m=n}^{N-1} \frac{n!}{\Gamma(n+\lambda+1)}\, z_{n+1,m+1}\, L_m^{\lambda}(t).$$

The quadrature formula for calculating the expansion coefficients $y(t)$ in matrix form is written as follows

$$Y_N = W_{N,N}\, YT_N, \quad (18)$$

where $W_{N,N}$ is a square matrix $N * N$ with elements $\omega_{i,j}$, and $YT_N$ is a column matrix with elements $y(t_j)$, $j = \overline{0, N-1}$.

If the values of the function $f(t)$ are calculated in points$t_k, k = \overline{1,K}$, then equality (17) will be in the matrix representation

$$\begin{pmatrix} f(t_1) \\ f(t_2) \\ \vdots \\ f(t_K) \end{pmatrix} = \begin{pmatrix} \eta_0(t_1) & \eta_1(t_1) & \dots & \eta_{N-1}(t_1) \\ \eta_0(t_2) & \eta_1(t_2) & \dots & \eta_{N-1}(t_2) \\ \vdots & \vdots & \ddots & \vdots \\ \eta_0(t_K) & \eta_1(t_K) & \dots & \eta_{N-1}(t_K) \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_K \end{pmatrix},$$

or

$$FT_K = \Theta_{K,N}\, Y_N. \quad (19)$$

Therefore, from formulas (17) and (18) we obtain the next formulas

$$FT_N = G_{K,N}\, YT_N, \qquad G_{K,N} = \Theta_{K,N}\, W_{N,N}. \quad (20)$$

Since the matrix $G_{K,N}$ is constructed, then equality (20) allows us to restore the values of the input signal based on the values of the output signal at the points $t_k$, $k = \overline{1,K}$.

The elements of the matrix do not depend on the values of the input and output signals. Therefore, it is possible to calculate their required number with specified accuracy by using specific software methods and storing them in the warehouse. This helps to save calculation time and reduce the accumulation of calculation errors.

## 6. Use of spectral methods in vegetation remote-sensing

The natural and physical phenomena investigation frequently is performed based on remote or spectroscopic data. This helps to obtain information on the studied objects without performing natural experiments, chemical or other analyses. The use of mathematical modeling allows us to save both financial and material costs, as well as research time.

Spectrometric surveys of objects using a remote method allow the detection of certain deviations in the spectra of the reflected signals from the background ones. Studying the difference between reflected and background signals makes it possible to identify these objects or detect the presence of impurities within them. Through remote spectrometric surveys, it is possible to detect certain deviations in the anomalous reflection spectra of plants from their background values [5, 6, 8, 18, 21]. Based on this, using the data of ground reconciliation and calibration works, the mapping of vegetation areas with anomalous spectral characteristics is easily performed. Effective methods for plant reflectance spectra processing, which must satisfy the given criteria, in particular, in terms of accuracy and extreme qualities should be developed to solve such type problems. The basis of Jacobi polynomials will be used to complete such problems. The input data for a specific interval of wavelengths is shown in Figure 2.

The reflected signal is approximated by the $N$ partial sum of the Fourier-Jacobi series

$$f(x) = \begin{Bmatrix} \omega(x) \\ 1 \end{Bmatrix} \sum_{n=0}^{\infty} \begin{Bmatrix} f_n \\ \hat{f}_n \end{Bmatrix} \frac{1}{r_n} P_n^{(\alpha,\beta)}(x), x \in [-1,1], \quad (21).$$

$$\begin{Bmatrix} f_n \\ \hat{f}_n \end{Bmatrix} = \int_{-1}^{1} \begin{Bmatrix} 1 \\ \omega(x) \end{Bmatrix} P_n^{(\alpha,\beta)}(x)\, f(x)\, dx.$$

It is advisable to use formula (8) to calculate the unknown coefficients $f_n$ as, during information processing, the values of the signals are known mainly at discrete points.

Contamination of plants by nitrates is known to cause disturbance of the reflected spectrum at wavelengths of 450 nm to 650 nm [5], and the number of nitrates may be determined after calibration by chemical samples for nitrate content by the deviation

$$\Delta f = f_{f\,max}(x_m) - f_{a\,max}(x_m)$$

at the maximum point $x = x_m$ (Fig. 1). Here, $f_f$ and $f_a$ are the value of the background and reflected spectra, respectively. Approximation of the reflected signal by various methods leads to the fact that the maximum point in the general case does not coincide with the background spectrum maximum point. This causes inaccuracy in the determination of the abnormal signal distribution $\Delta f$. Jacobi polynomials usage allows matching the coordinates of the background maximum and reflected spectra due to the presence of two free parameters $\alpha$ and $\beta$. Thus, the deviation of the reflected spectrum from the background spectrum is determined with specific accuracy.

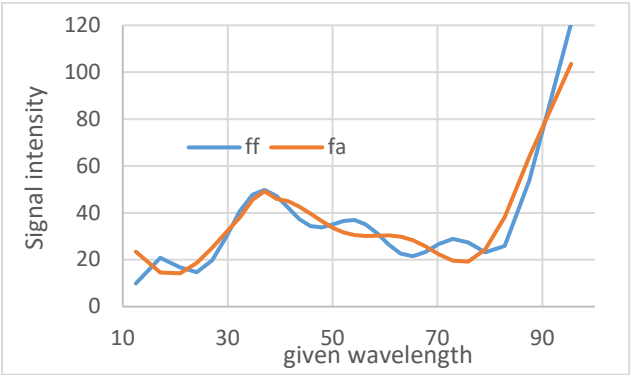


Figure 2: Dependence of background $f_f(\lambda_i)$ and reflected spectra on wavelength.

The identification algorithm is as follows.

a) The Fourier-Jacobi coefficients $f_{fn}$ and $f_{an}$ $(n = \overline{0, NV})$ are calculated based on the values of the $f_f(\lambda_i)$ background and $f_a(\lambda_i)$ reflected spectra.

b) The points of maximum $x_{fm}$ and $x_{am}$ for the background and reflected spectra at the given values $\alpha$ and $\beta$ in the interval 450-650 nm are obtained as solution of the below equations:

$$\sum_{n=0}^{NV} \frac{f_n}{r_n}\Big\{(\beta-\alpha-(\alpha+\beta)x)\,P_n^{(\alpha,\beta)}(x) + \frac{1}{2n+\alpha+\beta}\big((\alpha-\beta+(2n+\alpha+\beta)x)P_n^{(\alpha,\beta)}(x) - 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)\big)\Big\} = 0$$

and

$$\sum_{n=0}^{NV} \frac{\hat{f}_n}{r_n(2n+\alpha+\beta)}\Big\{n(\alpha-\beta-(2n+\alpha+\beta)x)P_n^{(\alpha,\beta)}(x) + 2(n+\alpha)(n+\beta)P_{n-1}^{(\alpha,\beta)}(x)\Big\} = 0.$$

The values of $\alpha$ and $\beta$ are chosen in such a way that equality takes place $x_m = x_{fm} = x_{am}$.

c) The values of the background $f_f$ and $\Delta f$ anomalous $f_a$ reflected spectra and their difference $\Delta f = f_a(x_m) - f_f(x_m)$ are calculated at the found maximum point. The values of $x_m$ and $\Delta f$ at different values of $\alpha$ and $\beta$ are presented in Table 1. The row marked with I corresponds to the case when the multiplier before the sum in equation (21) is equal to one, while the row marked with II - $\omega(x)$. The analysis of the data presented in Table 1 illustrates that for each $\alpha$ there are such values $\beta$, at which $\Delta f$ is constant. Persistence of disturbance of the background spectrum $\Delta f$ serves as a criterion for the optimal choice of parameters $\alpha$ and $\beta$ for the remote data spectrometry of plants under processing.

Table 1: The value of disturbance of the background spectrum $\Delta f$ of plant reflection at the $x_m$ maximum point for different $\alpha$ and $\beta$ parameters.

| $\beta$ | | -0.6 | | -0.28 | | 0.04 | | 0.36 | | 0.68 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\alpha$ | | $x_m$ | $\Delta f$ | $x_m$ | $\Delta f$ | $x_m$ | $\Delta f$ | $x_m$ | $\Delta f$ | $x_m$ | $\Delta f$ |
| -0.5 | I | 577.6 | 3.84 | 578.1 | 3.89 | 578.5 | 3.70 | 578.9 | 3.54 | 579.3 | 3.43 |
| | II | 543.8 | 6.18 | 596.9 | 3.53 | 543.7 | 6.33 | 545.6 | 6.58 | 533.0 | 9.87 |
| 0.0 | I | 577.4 | 4.17 | 577.9 | 3.97 | 578.3 | 3.77 | 578.7 | 3.59 | 579.1 | 3.46 |
| | II | 544.4 | 6.47 | 597.4 | 0.43 | 544.2 | 6.42 | 545.3 | 6.58 | 548.8 | 6.85 |
| 1.0 | I | 577.0 | 4.32 | 577.5 | 4.16 | 577.9 | 3.98 | 578.3 | 3.80 | 578.7 | 3.64 |
| | II | 570.9 | 7.16 | 569.8 | 6.73 | 529.5 | 5.98 | 529.2 | 5.90 | 529.1 | 5.84 |

The deviations of the reflected spectrum from the background obtained during the computational experiment were compared with the corresponding results obtained based on chemical analysis during the natural experiment. The convergence of results was within 15%. Therefore, the presence of nitrates in plants and the estimation of their percentage rate can be detected by the processing of remote data of green areas. Similarly, it is possible to detect the presence of heavy metals in green vegetation. For this purpose, it is necessary to investigate the deviation of the reflected spectrum from the background in the so-called "blue shift" zone, that is, at wavelengths longer than 2000 nm. In particular, this will allow determining the area of radioactive contamination spreading by remote methods without the risk of exposure to specialists.

## 7. Conclusions and discussion.

An algorithm for remote and spectroscopic information processing using the spectral method based on orthogonal Jacobi and Chebyshev-Laguerre polynomials is presented in the current paper. This algorithm allows approximating signals by orthogonal series, calculating the derivatives of the specified series, and solving approximation and identification problems on this basis. The feasibility and effectiveness of orthogonal Jacobi and Chebyshev-Laguerre polynomials used to solve signal-processing problems have been confirmed based on real-life data. The problem of the summation of the orthogonal basis series is incorrect. The optimal choice of free parameters makes it possible to provide an "implicit regularisation" by applying the fast convergence series. Depending on the specifics of the problem, the procedures for speeding up the convergence of series or modifying the functions with orthogonal series may be utilized for this purpose.

Based on the input information, the Fourier-Jacobi series can be presented in the following form:

$$\varphi(x) = \omega(x)\sum_{n=0}^{\infty}\frac{\varphi_{\omega n}}{r_n}P_n^{(\alpha,\beta)}(x),$$

$$\varphi(x) = (1-x)^{\alpha}\sum_{n=0}^{\infty}\frac{\varphi_{\alpha n}}{r_n}P_n^{(\alpha,\beta)}(x),$$

$$\varphi(x) = (1+x)^{\beta}\sum_{n=0}^{\infty}\frac{\varphi_{\beta n}}{r_n}P_n^{(\alpha,\beta)}(x).$$

A similar procedure can be provided with the Fourier-Laguerre series.

One of the reasons for the usage of classical orthogonal polynomials widespread is that they are easy to use and theoretically justified. Recently, intensive research has been done regarding constructing and applying biorthogonal polynomials to solve applied problems.

Algorithms for filtering one-dimensional signals in linear filter models were built based on one-dimensional Laguerre polynomials. One of the advantages here is that the integral convolution moves directly to series convolution without using the discretization procedure. In practice, it is necessary to consider models of filtering

signals of higher dimensions, particularly two-dimensional ones. In such cases, it is advisable to use multidimensional orthogonal Laguerre polynomials. Studies of these polynomials types were conducted in works [15, 16, 19]. The results obtained in these works can be successfully used for filtering multidimensional signals. Note that in practice, as a rule, signal filtering models are non-linear. However, a considerable number of signal filtering problems in nonlinear models can be solved with sufficient accuracy for practice by approximating them with linear filter models.

## Conflict of Interest

The authors declare no conflict of interest.

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